THE ABEL–JACOBI MAP FOR A CUBIC THREEFOLD AND PERIODS OF FANO THREEFOLDS OF DEGREE 14

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Abstract. The Abel–Jacobi maps of the families of elliptic quintics and rational quartics lying on a smooth cubic threefold are studied. It is proved that their generic fiber is the 5-dimensional projective space for quintics, and a smooth 3-dimensional variety birational to the cubic itself for quartics. The paper is a continuation of the recent work of Markushevich–Tikhomirov, who showed that the first Abel–Jacobi map factors through the moduli component of stable rank 2 vector bundles on the cubic threefold with Chern numbers $c_1 = 0, c_2 = 2$ obtained by Serre's construction from elliptic quintics, and that the factorizing map from the moduli space to the intermediate Jacobian is étale. The above result implies that the degree of the étale map is 1, hence the moduli component of vector bundles is birational to the intermediate Jacobian. As an applicaton, it is shown that the generic fiber of the period map of Fano varieties of degree 14 is birational to the intermediate Jacobian of the associated cubic threefold.

Introduction

Clemens and Griffiths studied in [CG] the Abel–Jacobi map of the family of lines on a cubic threefold X. They represented its intermediate Jacobian $J^2(X)$ as the Albanese variety $\mathrm{Alb}\, F(X)$ of the Fano surface $F(X)$ parametrizing lines on X and described its theta divisor. From this description, they deduced the Torelli Theorem and the nonrationality of X. Similar results were obtained by Tyurin [Tyu] and Beauville [B].

One can easily understand the structure of the Abel–Jacobi maps of some other familes of curves of low degree on X (conics, cubics or elliptic quartics), in reducing the problem to the results of Clemens–Griffiths and Tyurin. The first non trivial cases are those of rational normal quartics and of elliptic normal quintics. We determine the fibers of the Abel–Jacobi maps of these families of curves, in continuing the work started in [MT].

1991 *Mathematics Subject Classification.* 14J30,14J60,14J45.

Our result on elliptic quintics implies that the moduli space of instanton vector bundles of charge 2 on X has a component, birational to $J^2(X)$. We conjecture that the moduli space is irreducible, but the problem of irreducibility stays beyond the scope of the present article. As far as we know, this is the first example of a moduli space of vector bundles which is birational to an abelian variety, different from the Picard or Albanese variety of the base. The situation is also quite different from the known cases where the base is $\mathbb{P}^3$ or the 3-dimensional quadric. In these cases, the instanton moduli space is irreducible and rational at least for small charges, see [Barth], [ES], [H], [LP], [OS]. Remark, that for the cubic X, two is the smallest possible charge, but the moduli space is non-rational. There are no papers on the geometry of particular moduli spaces of vector bundles for other 3-dimensional Fano varieties (for some constructions of vector bundles on such varieties, see [G1], [G2], [B-MR1], [B-MR2],[SW], [AC]).

The authors of [MT] proved that the Abel–Jacobi map Φ of the family of elliptic quintics lying on a general cubic threefold X factors through a 5-dimensional moduli component M_X of stable rank 2 vector bundles $\mathcal{E}$ on X with Chern numbers $c_1 = 0, c_2 = 2$. The factorizing map ϕ sends an elliptic quintic $C \subset X$ to the vector bundle $\mathcal{E}$ obtained by Serre's construction from C (see Sect. 2). The fiber $\phi^{-1}([\mathcal{E}])$ is a 5-dimensional projective space in the Hilbert scheme Hilb_X^{5n}, and the map Ψ from the moduli space to the intermediate Jacobian $J^2(X)$, defined by $\Phi = \Psi \circ \phi$, is étale on the open set representing (smooth) elliptic quintics which are not contained in a hyperplane (Theorem 2.1).

We improve the result of [MT] in showing that the degree of the above étale map is 1. Hence M_X is birational to $J^2(X)$ and the generic fiber of Φ is just one copy of $\mathbb{P}^5$ (see Theorem 3.2 and Corollary 3.3). The behavior of the Abel–Jacobi map of elliptic quintics is thus quite similar to that of the Abel–Jacobi map of divisors on a curve, where all the fibers are projective spaces. But we prove that the situation is very different in the case of rational normal quartics, where the fiber of the Abel–Jacobi map is a *non-rational* 3-dimensional variety: it is birationally equivalent to the cubic X itself (Theorem 5.2).

The first new ingredient of our proofs, comparing to [MT], is another interpretation of the vector bundles $\mathcal{E}$ from M_X. We represent the cubic X as a linear section of the Pfaffian cubic in $\mathbb{P}^{14}$, parametrizing 6×6 matrices M of rank 4, and realize $\mathcal{E}^\vee(-1)$ as the restriction of the kernel bundle $M \mapsto \ker M \subset \mathbb{C}^6$ (Theorem 2.2). The kernel bundle has been investigated by A. Adler in his Appendix to [AR]. We prove that it embeds X into the Grassmannian $G = G(2, 6)$, and the quintics $C \in \phi^{-1}([\mathcal{E}])$ become the sections of X by the Schubert varieties $\sigma_{11}(L)$ for

all hyperplanes $L \subset \mathbb{C}^6$. We deduce that for any line $l \subset X$, each fiber of ϕ contains precisely one pencil $\mathbb{P}^1$ of reducible curves of the form $C'+l$ (Lemma 3.4). Next we use the techniques of Hartshorne–Hirschowitz [HH] for smoothing the curves of the type "a rational normal quartic plus one of its chords in X" (see Sect. 4) to show that there is a 3-dimensional family of such curves in a generic fiber of ϕ and that the above pencil $\mathbb{P}^1$ for a generic l contains curves $C' + l$ of this type (Lemma 4.3, Corollary 4.4).

The other main ingredient is the parametrization of $J^2(X)$ by minimal sections of the 2-dimensional conic bundles of the form $Y(C^2) = \pi_l^{-1}(C^2)$, where $\pi_l : \mathrm{Blowup}_l(X) \longrightarrow \mathbb{P}^2$ is the conic bundle obtained by projecting X from a fixed line l, and C^2 is a generic conic in $\mathbb{P}^2$ (see Sect. 3). The standard Wirtinger approach [B] parametrizes $J^2(X)$ by reducible curves which are sums of components of reducible fibers of π_l. Our approach, developed in [I] in a more general form, replaces the degree 10 sums of components of the reducible fibers of the surfaces $Y(C^2)$ by the irreducible curves which are sections of the projection $Y(C^2) \longrightarrow C^2$ with a certain minimality condition. This gives a parametrization of $J^2(X)$ by a family of rational curves, each one of which is projected isomorphically onto some conic in $\mathbb{P}^2$. It turns out, that these rational curves are normal quartics meeting l at two points. They form a *unique* pencil $\mathbb{P}^1$ in each fiber of the Abel–Jacobi map of rational normal quartics. Combining this with the above, we conclude that the curves of type $C' + l$ form a unique pencil in each fiber of Φ, hence the fiber is one copy of $\mathbb{P}^5$.

In conclusion, we provide a description of the moduli space of Fano varieties V_{14} as a birationally fibered space over the moduli space of cubic 3-folds with the intermediate Jacobian as a fiber (see Theorem 5.8). The interplay between cubics and varieites V_{14} is exploited several times in the paper. We use the Fano–Iskovskikh birationality between X and V_{14} to prove Theorem 2.2 on kernel bundles, and the Tregub–Takeuchi one (see Sect. 1) to study the fiber of the Abel–Jacobi map of the family of rational quartics (Theorem 5.2) and the relation of this family to that of normal elliptic quintics (Proposition 5.6).

Acknowledgements. The second author acknowledges with pleasure the hospitality of the MPIM at Bonn, where he completed the work on the paper.

1. Birational isomorphisms between V_3 and V_{14}

There are two constructions of birational isomorphisms between a nonsingular cubic threefold $V_3 \in \mathbb{P}^4$ and the Fano variety V_{14} of degree

14 and of index 1, which is a nonsingular section of the Grassmannian $G(2,6) \in \mathbb{P}^{14}$ by a linear subspace of codimension 5. The first one is that of Fano–Iskovskikh, and it gives a birational isomorphism whose indeterminacy locus in both varieties is an elliptic curve together with some 25 lines; the other is due to Tregub–Takeuchi, and its indeterminacy locus is a rational quartic plus 16 lines on the side of V_3, and 16 conics passing through one point on the side of V_{14}. We will sketch both of them.

Theorem 1.1 (Fano–Iskovskikh)**.** *Let $X = V_3$ be a smooth cubic threefold. Then X contains a smooth projectively normal elliptic quintic curve. Let C be such a curve. Then C has exactly 25 bisecant lines $l_i \subset X$, $i = 1, ..., 25$, and there is a unique effective divisor $M \in \mid \mathcal{O}_X(5 - 3C) \mid$ on X, which is a reduced surface containing the l_i. The following assertions hold:*

(i) The non-complete linear system $\mid \mathcal{O}_X(7-4C) \mid$ defines a birational map $\rho : X \to V$ where $V = V_{14}$ is a Fano 3-fold of index 1 and of degree 14. Moreover $\rho = \sigma \circ \kappa \circ \tau$ where $\sigma : X' \to X$ is the blow-up of C, $\kappa : X' \to X^+$ is a flop over the proper transforms $l'_i \subset X'$ of the l_i, $i = 1, ..., 25$, and $\tau : X^+ \to V$ is a blowdown of the proper transform $M^+ \subset X^+$ of M onto an elliptic quintic $B \subset V$. The map τ sends the transforms $l_i^+ \subset X^+$ of l_i to the 25 secant lines $m_i \subset V$, $i = 1, ..., 25$ of the curve B.

(ii) The inverse map ρ^{-1} is defined by the system $\mid \mathcal{O}_V(3 - 4B) \mid$. The exceptional divisor $E' = \sigma^{-1}(C) \subset X'$ is the proper transform of the unique effective divisor $N \in \mid \mathcal{O}_V(2 - 3B) \mid$.

For a proof , see [Isk1], [F], or [Isk-P], Ch. 4.

Theorem 1.2 (Tregub–Takeuchi)**.** *Let X be a smooth cubic threefold. Then X contains a rational projectively normal quartic curve. Let Γ be such a curve. Then Γ has exactly 16 bisecant lines $l_i \subset X$, $i = 1, ..., 16$, and there is a unique effective divisor $M \in \mid \mathcal{O}_X(3-2\Gamma) \mid$ on X, which is a reduced surface containing the l_i. The following assertions hold:*

(i) The non-complete linear system $\mid \mathcal{O}_X(8-5\Gamma) \mid$ defines a birational map $\chi : X \to V$ where V is a Fano 3-fold of index 1 and of degree 14. Moreover $\chi = \sigma \circ \kappa \circ \tau$, where $\sigma : X' \to X$ is the blowup of Γ, $\kappa : X' \to X^+$ is a flop over the proper transforms $l'_i \subset X'$ of l_i, $i = 1, ..., 16$, and $\tau : X^+ \to V$ is a blowdown of the proper transform $M^+ \subset X^+$ of M to a point $P \in V$. The map τ sends the transforms $l_i^+ \subset X^+$ of l_i to the 16 conics $q_i \subset V$, $i = 1, ..., 16$ which pass through the point P.

(ii) The inverse map χ^{-1} is defined by the system $\mid \mathcal{O}_V(2-5P) \mid$. The exceptional divisor $E' = \sigma^{-1}(\Gamma) \subset X'$ is the proper transform of the unique effective divisor $N \in\mid \mathcal{O}_V(3-8P) \mid$.

(iii) For a generic point P on any nonsingular V_{14}, this linear system defines a birational isomorphism of type χ^{-1}.

Proof. For (i), (ii), see [Tak], Theorem 3.1, and [Tre]. For (iii), see [Tak], Theorem 2.1, (iv). See also [Isk-P], Ch. 4. □

1.3. **Geometric description.** We will briefly describe the geometry of the first birational isomorphism between V_3 and V_{14} following [P].

Let E be a 6-dimensional vector space over $\mathbb{C}$. Fix a basis $e_0, \ldots, e_5$ for E, then $e_i \wedge e_j$ for $0 \leq i < j \leq 5$ form a basis for the Plücker space of 2-spaces in E, or equivalently, of lines in $\mathbb{P}^5 = \mathbb{P}(E)$. With Plücker coordinates x_{ij}, the embedding of the Grassmannian $G = G(2,E)$ in $\mathbb{P}^{14} = \mathbb{P}(\wedge^2 E)$ is precisely the locus of rank 2 skew symmetric 6×6 matrices

$$M = \begin{bmatrix} 0 & x_{01} & x_{02} & x_{03} & x_{04} & x_{05} \\ -x_{01} & 0 & x_{12} & x_{13} & x_{14} & x_{15} \\ -x_{02} & -x_{12} & 0 & x_{23} & x_{24} & x_{25} \\ -x_{03} & -x_{13} & -x_{23} & 0 & x_{34} & x_{35} \\ -x_{04} & -x_{14} & -x_{24} & -x_{34} & 0 & x_{45} \\ -x_{05} & -x_{15} & -x_{25} & -x_{35} & -x_{45} & 0 \end{bmatrix}.$$

There are two ways to associate to these data a 13-dimensional cubic. The Pfaffian cubic hypersurface $\Xi \subset \mathbb{P}^{14}$ is defined as the zero locus of the 6×6 Pfaffian of this matrix; it can be identified with the secant variety of $G(2,E)$, or else, it is the locus where M has rank 4. The other way is to consider the dual variety $\Xi' = G^\vee \subset \mathbb{P}^{14\vee}$ of G; it is also a cubic hypersurface, which is nothing other than the secant variety of the Grassmannian $G' = G(2, E^\vee) \subset \mathbb{P}(\wedge^2 E^\vee) = \mathbb{P}^{14\vee}$.

As it is classically known, the generic cubic threefold X can be represented as a section of the Pfaffian cubic by a linear subspace of codimension 10; see also a recent proof in [AR], Theorem 47.3. There are ∞^5 essentially different ways to do this. Beauville and Donagi [BD] have used this idea for introducing the symplectic structure on the Fano 4-fold (parametrizing lines) of a cubic 4-fold. In their case, only special cubics (a divisorial family) are sections of the Pfaffian cubic, so they introduced the symplectic structure on the Fano 4-folds of these special cubics, and obtained the existence of such a structure on the generic one by deformation arguments.

For any hyperplane section $H \cap G$ of G, we can define $\operatorname{rk} H$ as the rank of the antisymmetric matrix (α_{ij}), where $\sum \alpha_{ij} x_{ij} = 0$ is the

equation of H. So, $\operatorname{rk} H$ may take the values 2,4 or 6. If $\operatorname{rk} H = 6$, then $H \cap G$ is nonsingular and for any $p \in \mathbb{P}^5 = \mathbb{P}(E)$, there is the unique hyperplane $L_p \subset \mathbb{P}^5 = \mathbb{P}(E)$, such that $q \in H \cap G$, $p \in l_q \iff l_q \subset L_p$. Here l_q denotes the line in $\mathbb{P}^5$ represented by $q \in G$. (This is a way to see that the base of the family of 3-dimensional planes on the 7-fold $H \cap G$ is $\mathbb{P}^5$.)

The rank of H is 4 if and only if H is tangent to G at exactly one point z, and in this case, the hyperplane L_p is not defined for any $p \in l_z$: we have for such p the equivalence $p \in l_x \iff x \in H$. Following Puts, we call the line l_z the *center* of H; it will be denoted c_H.

In the third case, when $\operatorname{rk} H = 2$, $H \cap G$ is singular along the whole Grassmannian subvariety $G(2,4) = G(2, E_H)$, where $E_H = \ker(\alpha_{ij})$ is of dimension 4. We have $x \in H \iff l_x \cap \mathbb{P}(E_H) \neq \varnothing$.

This description identifies the dual of G with $\Xi' = \{H \mid \operatorname{rk} H \leq 4\} = \{H \mid Pf((\alpha_{ij})) = 0$, and its singular locus with $\{E_H\}_{\operatorname{rk} H=2} = G(4, E)$.

Now, associate to any nonsingular $V_{14} = G \cap \Lambda$, where $\Lambda = H_1 \cap H_2 \cap H_3 \cap H_4 \cap H_5$, the cubic 3-fold V_3 by the following rule:

$$V_{14} = G \cap \Lambda \ \mapsto \ V_3 = \Xi' \cap \Lambda^\vee, \tag{1}$$

where $\Lambda^\vee =< H_1^\vee, H_2^\vee, H_3^\vee, H_4^\vee, H_5^\vee >$, $H_i^\vee$ denotes the orthogonal complement of H_i in $\mathbb{P}^{14\vee}$, and the angular brackets the linear span. One can prove that V_3 is also nonsingular.

According to Fano, the lines l_x represented by points $x \in V_{14}$ sweep out an irreducible quartic hypersurface W, which Fano calls the quartic da Palatin. W coincides with the union of centers of all $H \in V_3$. One can see, that W is singular along the locus of foci p of Schubert pencils of lines on G

$$\sigma_{43}(p, h) = \{x \in G \mid p \in l_x \subset h\}$$

which lie entirely in V_{14}, where h denotes a plane in $\mathbb{P}^5$ (depending on p). The pencils σ_{43} are exactly the lines on V_{14}, so $\operatorname{Sing} W$ is identified with the base of the family of lines on V_{14}, which is known to be a nonsingular curve of genus 26 for generic V_{14} (see, e. g. [M] for the study of the curve of lines on V_{14}, and Sections 50, 51 of [AR] for the study of $\operatorname{Sing} W$ without any connection to V_{14}).

The construction of the birational isomorphism $\eta_L : V_{14} \dashrightarrow V_3$ depends on the choice of a hyperplane $L \subset \mathbb{P}^5$. Let

$$\phi : V_{14} \dashrightarrow W \cap L, \ x \mapsto L \cap l_x \ , \psi : V_3 \dashrightarrow W \cap L, \ H^\vee \mapsto L \cap c_H.$$

These two maps are birational, and η_L is defined by

$$\eta_L = \psi^{-1} \circ \phi. \tag{2}$$

The locus, on which η_L is not an isomorphism, consists of points where either ϕ or ψ is not defined or is not one-to-one. The indeterminacy locus B of ϕ consists of all the points x such that $l_x \subset L$, that is, $B = G(2, L) \cap H_1 \cap \ldots \cap H_5$. For generic L, it is obviously a smooth elliptic quintic curve in V_{14}, and it is this curve that was denoted in Theorem 1.1 by the same symbol B. The indeterminacy locus of ψ is described in a similar way. We summarize the above in the following statement.

Proposition 1.4. *Any nonsingular variety V_{14} determines a unique nonsingular cubic V_3 by the rule (1). Conversely, a generic cubic V_3 can be obtained in this way from ∞^5 many varieties V_{14}.*

For each pair (V_{14}, V_3) related by (1), there is a family of birational maps $\eta_L : V_{14} \dashrightarrow V_3$, defined by (2) and parametrized by points of the dual projective space $\mathbb{P}^{5\vee}$, and the structure of η_L for generic L is described by Theorem 1.1.

The smooth elliptic quintic curve B (resp. C) of Theorem 1.1 is the locus of points $x \in V_{14}$ such that $l_x \subset L$ (resp. $H^\vee \in V_3$ such that $c_H \subset L$).

Definition 1.5. We will call two varieties V_3, V_{14} associated (to each other), if V_3 can be obtained from V_{14} by the construction (1).

1.6. **Intermediate Jacobians of V_3, V_{14}.** Both constructions of birational isomorphisms give the isomorphism of the intermediate Jacobians of generic varieties V_3, V_{14}, associated to each other. This is completely obvious for the second construction: it gives a birational isomorphism, which is a composition of blowups and blowdowns with centers in nonsingular rational curves or points. According to [CG], a blowup $\sigma : \tilde{X} \longrightarrow X$ of a threefold X with a nonsingular center Z can change its intermediate Jacobian only in the case when Z is a curve of genus ≥ 1, and in this case $J^2(\tilde{X}) \simeq J^2(X) \times J(Z)$ as principally polarized abelian varieties, where J^2 (resp. J) stands for the intermediate Jacobian of a threefold (resp. for the Jacobian of a curve). Thus, the Tregub–Takeuchi birational isomorphism does not change the intermediate Jacobian. Similar argument works for the Fano–Iskovskikh construction. It factors through blowups and blowdowns with centers in rational curves, and contains in its factorization exactly one blowup and one blowdown with nonrational centers, which are elliptic curves. So, we have $J^2(V_3) \times C \simeq J^2(V_{14}) \times B$ for some elliptic curves C, B. According to Clemens–Griffiths, $J^2(V_3)$ is irreducible for every nonsingular V_3, so we can simplify the above isomorphism to obtain

$J^2(V_3) \simeq J^2(V_{14})$ (and we also obtain, as a by-product, the isomorphism $C \simeq B$).

Proposition 1.7. *Let $V = V_{14}$, $X = V_3$ be a pair of smooth Fano varieties related by either of the two birational isomorphisms of Fano–Iskovskikh or of Tregub–Takeuchi. Then $J^2(X) \simeq J^2(V)$, V, X are associated to each other and related by a birational isomorphism of the other type as well.*

Proof. The isomorphism of the intermediate Jacobians was proved in the previous paragraph. Let $J^2(V') = J^2(V'') = J$. By Clemens-Griffiths [CG] or Tyurin [Tyu], the global Torelli Theorem holds for smooth 3-dimensional cubics, so there exists the unique cubic threefold X such that $J^2(X) = J$ as p.p.a.v. Let X' and X'' be the unique cubics associated to V' and V''. Since $J^2(X') = J^2(V') = J = J^2(V'') = J^2(X'')$, then $X' \simeq X \simeq X''$.

Let now V' and V'' be associated to the same cubic threefold X, and let $J^2(X) = J$. Then by the above $J^2(V') = J^2(X) = J^2(V'')$.

Let X, V be related by, say, a Tregub–Takeuchi birational isomorphism. By Proposition 1.4, V contains a smooth elliptiic quintic curve and admits a birational isomorphism of Fano–Iskovskikh type with some cubic X'. Then, as above, $X \simeq X'$ by Global Torelli, and X, V are associated to each other by the definition of the Fano–Iskovskikh birational isomorphism. Conversely, if we start from the hypothesis that X, V are related by a Fano–Iskovskikh birational isomorphism, then the existence of a Tregub–Takeuchi one from V to some cubic X' is affirmed by Theorem 1.2, (iii). Hence, again by Global Torelli, $X \simeq X'$ and we are done.

$\square$

2. Abel–Jacobi map and vector bundles on a cubic threefold

Let X be a smooth cubic threefold. The authors of [MT] have associated to every normal elliptic quintic curve $C \subset X$ a stable rank 2 vector bundle $\mathcal{E} = \mathcal{E}_C$, unique up to isomorpfism. It is defined by Serre's construction:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E}(1) \longrightarrow \mathcal{I}_C(2) \longrightarrow 0 \ , \tag{3}$$

where $\mathcal{I}_C = \mathcal{I}_{C,X}$ is the ideal sheaf of C in X. Since the class of C modulo algebraic equivalence is $5l$, where l is the class of a line, the sequence (3) implies that $c_1(\mathcal{E}) = 0, c_2(\mathcal{E}) = 2l$. One sees immediately from (3) that $\det \mathcal{E}$ is trivial, and hence $\mathcal{E}$ is self-dual as soon as it is a

vector bundle (that is, $\mathcal{E}^\vee \simeq \mathcal{E}$). See [MT, Sect. 2] for further details on this construction.

Let $\mathcal{H}^* \subset \mathrm{Hilb}^{5n}_X$ be the open set of the Hilbert scheme parametrizing normal elliptic quintic curves in X, and $M \subset M_X(2;0,2)$ the open subset in the moduli space of vector bundles on X parametrizing those stable rank 2 vector bundles which arise via Serre's construction from normal elliptic quintic curves. Let $\phi^* : \mathcal{H}^* \longrightarrow M$ be the natural map. For any reference curve C_0 of degree 5 in X, let $\Phi^* : \mathcal{H}^* \longrightarrow J^2(X)$, $[C] \mapsto [C - C_0]$, be the Abel–Jacobi map. The following result is proved in [MT].

Theorem 2.1. *$\mathcal{H}^*$ and M are smooth of dimensions 10 and 5 respectively. They are also irreducible for generic X. There exist a bigger open subset $\mathcal{H} \subset \mathrm{Hilb}^{5n}_X$ in the nonsingular locus of Hilb^{5n}_X containing $\mathcal{H}^*$ as a dense subset and extensions of ϕ^*, Φ^* to morphisms ϕ, Φ respectively, defined on the whole of $\mathcal{H}$, such that the following properties are verified:*

(i) ϕ is a locally trivial fiber bundle in the étale topology with fiber $\mathbb{P}^5$. For every $[\mathcal{E}] \in M$, we have $h^0(\mathcal{E}(1)) = 6$, and $\phi^{-1}([\mathcal{E}]) \subset \mathcal{H}$ is nothing but the $\mathbb{P}^5$ of zero loci of all the sections of $\mathcal{E}(1)$.

(ii) The fibers of Φ are finite unions of those of ϕ, and the map $\Psi : M \longrightarrow J^2(X)$ in the natural factorization $\Phi = \Psi \circ \phi$ is a quasi-finite étale morphism.

Now, we will give another interpretation of the vector bundles $\mathcal{E}_C$. Let us represent the cubic $X = V_3$ as a section of the Pfaffian cubic $\Xi' \subset \mathbb{P}^{14\vee}$ and keep the notation of 1.3. Let $\mathcal{K}$ be the kernel bundle on X whose fiber at $M \in X$ is $\ker H$. Thus $\mathcal{K}$ is a rank 2 vector subbundle of the trivial rank 6 vector bundle $E_X = E \otimes_{\mathbb{C}} \mathcal{O}_X$. Let $i : X \longrightarrow \mathbb{P}^{14}$ be the composition $\mathrm{Pl} \circ \mathrm{Cl}$, where $\mathrm{Cl} : X \longrightarrow G(2, E)$ is the classifying map of $\mathcal{K} \subset E_X$, and $\mathrm{Pl} : G(2, E) \hookrightarrow \mathbb{P}(\wedge^2 E) = \mathbb{P}^{14}$ the Plücker embedding.

Theorem 2.2. *For any vector bundle $\mathcal{E}$ obtained by Serre's construction starting from a normal elliptic quintic $C \subset X$, there exists a representation of X as a linear section of Ξ' such that $\mathcal{E}(1) \simeq \mathcal{K}^\vee$ and all the global sections of $\mathcal{E}(1)$ are the images of the constant sections of $E^\vee_X$ via the natural map $E^\vee_X \longrightarrow \mathcal{K}^\vee$. For generic $X, \mathcal{E}$, such a representation is unique modulo the action of $PGL(6)$ and the map i can be identified with the restriction $v_2|_X$ of the Veronese embedding $v_2 : \mathbb{P}^4 \longrightarrow \mathbb{P}^{14}$ of degree 2.*

Proof. Let $C \subset X$ be a normal elliptic quintic. By Theorem 1.1, there exists a $V_{14} = G \cap \Lambda$ together with a birational isomorphism $X \dashrightarrow V_{14}$. Proposition 1.7 implies that X and V_{14} are associated to each other.

By Proposition 1.4, we have $C = \{H^\vee \in X \mid c_H \subset L\} = \mathrm{Cl}^{-1}(\sigma_{11}(L))$, where $\sigma_{11}(L)$ denotes the Schubert variety in G parametrizing the lines $c \subset \mathbb{P}(E)$ contained in L. It is standard that $\sigma_{11}(L)$ is the scheme of zeros of a section of the dualized universal rank 2 vector bundle $\mathcal{S}^\vee$ on G. Hence C is the scheme of zeros of a section of $\mathcal{K}^\vee = \mathrm{Cl}^*(\mathcal{S}^\vee)$. Hence $\mathcal{K}^\vee$ can be obtained by Serre's construction from C, and by uniqueness, $\mathcal{K}^\vee \simeq \mathcal{E}_C(1)$.

By Lemma 2.1, c) of [MT], $h^0(\mathcal{E}_C(1)) = 6$, so, to prove the assertion about global sections, it is enough to show the injectivity of the natural map $E^\vee = H^0(E_X^\vee) \longrightarrow H^0(\mathcal{K}^\vee)$. The latter is obvious, because the quartic da Palatini is not contained in a hyperplane. Thus we have $E^\vee = H^0(\mathcal{K}^\vee)$.

For the identification of i with $v_2|_X$, it is sufficient to show that i is defined by the sections of $\mathcal{O}(2)$ in the image of the map ev : $\Lambda^2 H^0(\mathcal{E}(1)) \longrightarrow H^0(\det(\mathcal{E}(1))) = H^0(\mathcal{O}(2))$ and that ev is an isomorphism. This is proved in the next lemmas. The uniqueness modulo $PGL(6)$ is proved in Lemma 2.7.

$\square$

Lemma 2.3. *Let* $\mathrm{Pf}_2 : \mathbb{P}^{14} \dashrightarrow \mathbb{P}^{14}$ *be the Pfaffian map, sending a skew-symmetric* 6×6 *matrix* M *to the collection of its 15 quadratic Pfaffians. Then* $\mathrm{Pf}_2^2 = \mathrm{id}_{\mathbb{P}^{14}}$*, the restriction of* Pf_2 *to* $\mathbb{P}^{14} \setminus \Xi$ *is an isomorphism onto* $\mathbb{P}^{14} \setminus G$*, and* $i = \mathrm{Pf}_2|_X$*.*

Thus Pf_2 is an example of a Cremona quadratic transformation. Such transformations were studied in [E-SB].

Proof. Let (e_i), (ϵ_i) be dual bases of $E, E^\vee$ respectively, and $(e_{ij} = e_i \wedge e_j), (\epsilon_{ij})$ the corresponding bases of $\wedge^2 E$, $\wedge^2 E^\vee$. Identify M in the source of Pf_2 with a 2-form $M = \sum a_{ij}\epsilon_{ij}$. Then Pf_2 can be given by the formula $\mathrm{Pf}_2(M) = \frac{1}{2!4!} M \wedge M \llcorner e_{123456}$, where $e_{123456} = e_1 \wedge \ldots \wedge e_6$, and $\llcorner$ stands for the contraction of tensors. Notice that Pf_2 sends 2-forms of rank 6,4, resp. 2 to bivectors of rank 6,2, resp. 0. Hence Pf_2 is not defined on G' and contracts $\Xi' \setminus G'$ into G. In fact, the Pfaffians of a 2-form M of rank 4 are exactly the Plücker coordinates of $\ker M$, which implies $i = \mathrm{Pf}_2|_X$.

In order to iterate Pf_2, we have to identify its source $\mathbb{P}(\wedge^2 E^\vee)$ with its target $\mathbb{P}(\wedge^2 E)$. We do it in using the above bases: $\epsilon_{ij} \mapsto e_{ij}$. Let $N = \mathrm{Pf}_2^2(M) = \sum b_{ij}\epsilon_{ij}$. Then each matrix element $b_{ij} = b_{ij}(M)$ is a polynomial of degree 4 in (a_{kl}), vanishing on Ξ'. Hence it is divisible by the equation of Ξ', which is the cubic Pfaffian $\mathrm{Pf}(M)$. We can write $b_{ij} = \tilde{b}_{ij}\,\mathrm{Pf}(M)$, where $\tilde{b}_{ij}$ are some linear forms in (a_{kl}). Testing them on a collection of simple matrices with only one variable matrix

element, we find the answer: $\mathrm{Pf}_2(M) = \mathrm{Pf}(M)M$. Hence Pf_2 is a birational involution. $\square$

Lemma 2.4. *Let $l \subset V_3$ be a line. Then $i(l)$ is a conic in $\mathbb{P}^{14}$, and the lines of $\mathbb{P}^5$ parametrized by the points of $i(l)$ sweep out a quadric surface of rank 3 or 4.*

Proof. The restriction of Cl to the lines in V_3 is written out in [AR] on pages 170 (for a non-jumping line of $\mathcal{K}$, formula (49.5)) and 171 (for a jumping line). These formulas imply the assertion; in fact, the quadric surface has rank 4 for a non-jumping line, and rank 3 for a jumping one. $\square$

Lemma 2.5. *The map i is injective.*

Proof. Let $\tilde{\Xi}$ be the natural desingularization of Ξ' parametrizing pairs (M, l), where M is a skew-symmetric 6×6 matrix and l is a line in the projectivized kernel of M. We have $\tilde{\Xi} = \mathbb{P}(\wedge^2(E_X/\mathcal{S}))$, where $\mathcal{S}$ is the tautological rank 2 vector bundle on $G = G(2,6)$. $\tilde{\Xi}$ has two natural projections $p : \tilde{\Xi} \longrightarrow G \subset \mathbb{P}^{14}$ and $q : \tilde{\Xi} \longrightarrow \Xi' \subset \mathbb{P}^{14\vee}$. The classifying map of $\mathcal{K}$ is just $\mathrm{Cl} = pq^{-1}$. q is isomorphic over the alternating forms of rank 4, so $q^{-1}(V_3) \simeq V_3$. p is at least bijective on $q^{-1}(V_3)$. In fact, it is easy to see that the fibers of p can only be linear subspaces of $\mathbb{P}^{14}$. Indeed, the fiber of p is nothing but the family of matrices M whose kernel contains a fixed plane, hence it is a linear subspace $\mathbb{P}^5$ of $\mathbb{P}^{14\vee}$, and the fibers of $p|_{q^{-1}(V_3)}$ are $\mathbb{P}^5 \cap V_3$. As V_3 does not contain planes, the only possible fibers are points or lines. By the previous lemma, they can be only points, so i is injective. $\square$

Lemma 2.6. *i is defined by the image of the map* $\mathrm{ev} : \Lambda^2 H^0(\mathcal{E}(1)) \longrightarrow H^0(\det(\mathcal{E}(1))) = H^0(\mathcal{O}(2))$ *considered as a linear subsystem of* $|\mathcal{O}(2)|$.

Proof. Let $(x_i = \epsilon_i)$ be the coordinate functions on E, dual to the basis (e_i). The x_i can be considered as sections of $\mathcal{K}^\vee$. Then $x_i \wedge x_j$ can be considered either as an element x_{ij} of $\wedge^2 E^\vee = \wedge^2 H^0(\mathcal{K}^\vee)$, or as a section s_{ij} of $\wedge^2 \mathcal{K}^\vee$. For a point $x \in V_3$, the Plücker coordinates of the corresponding plane $K_x \subset E$ are $x_{ij}(\nu)$ for a non zero bivector $\nu \in \wedge^2 K_x$. By construction, this is the same as $s_{ij}(x)(\nu)$. This proves the assertion.

$\square$

Lemma 2.7. *Let $X \xrightarrow{\sim} \Xi' \cap \Lambda_1$, $X \xrightarrow{\sim} \Xi' \cap \Lambda_2$ be two representations of X as linear sections of Ξ', $\mathcal{K}_1, \mathcal{K}_2$ the corresponding kernel bundles on X. Assume that $\mathcal{K}_1 \simeq \mathcal{K}_2$. Then there exists a linear transformation*

$A \in GL(E^\vee) = GL_6$ such that $\Xi' \cap \wedge^2 A(\Lambda_1)$ and $\Xi' \cap \Lambda_2$ have the same image under the classifying maps into G. The family of linear sections $\Xi' \cap \Lambda$ of the Pfaffian cubic with the same image in G is a rationally 1-connected subvariety of $G(5, 15)$, generically of dimension 0.

Proof. The representations $X \tilde{\longrightarrow} \Xi' \cap \Lambda_1$, $X \tilde{\longrightarrow} \Xi' \cap \Lambda_2$ define two isomorphisms $f_1 : E^\vee \longrightarrow H^0(\mathcal{K}_1)$, $f_2 : E^\vee \longrightarrow H^0(\mathcal{K}_2)$. Identifying $\mathcal{K}_1, \mathcal{K}_2$, define $A = f_2^{-1} \circ f_1$.

Assume that $\Lambda = \wedge^2 A(\Lambda_1) \neq \Lambda_2$. Then the two 3-dimensional cubics $\Xi' \cap \Lambda$ and $\Xi' \cap \Lambda_2$ are isomorphic by virtue of the map $f = f_2 \circ f_1^{-1} \circ (\wedge^2 A)^{-1}$. By construction, we have $\ker M = \ker f(M)$ for any $M \in \Xi' \cap \Lambda$. Hence $\Xi' \cap \Lambda$ and $\Xi' \cap \Lambda_2$ represent two cross-sections of the map pq^{-1} defined in the proof of Lemma 2.5 over their common image $Y = pq^{-1}(\Xi' \cap \Lambda) = pq^{-1}(\Xi' \cap \Lambda_2)$, and f is a morphism over Y. These cross-sections do not meet the indeterminacy locus $G' \subset \Xi'$ of pq^{-1}, because it is at the same time the singular locus of Ξ' and both 3-dimensional cubics are nonsingular. The fibers of pq^{-1} being linear subspaces of $\mathbb{P}^{14\vee}$, the generic element of a linear pencil $X_{\lambda:\mu} = \Xi' \cap (\lambda\Lambda + \mu\Lambda_2)$ represents also a cross-section of pq^{-1} that does not meet G'. So there is a one-dimensional family of representations of X as a linear section of the Pfaffian cubic which are not equivalent under the action of $PGL(6)$ but induce the same vector bundle $\mathcal{K}$. This family joins $\Xi' \cap \Lambda$ and $\Xi' \cap \Lambda_2$ and its base is an open subset of $\mathbb{P}^1$. This cannot happen for generic $X, \mathcal{E}$, because both the family of vector bundles $\mathcal{E}$ and that of representations of X as a linear section of Ξ' are 5 dimensional for generic X (Theorem 2.1 and Proposition 1.4).

$\square$

Lemma 2.8. *For a generic 3-dimensional linear section V_3 of Ξ', the 15 quadratic Pfaffians of $M \in V_3$ are linearly independent in $|\mathcal{O}_{V_3}(2)|$.*

The authors of [IR] solve a similar problem: they describe the structure of the restriction of Pf_2 to a 4-dimensional linear section of the Pfaffian cubic.

Proof. It is sufficient to verify this property for a special V_3. Take Klein's cubic

$$v^2 w + w^2 x + x^2 y + y^2 z + z^2 v = 0.$$

Adler ([AR], Lemma (47.2)) gives the representation of this cubic as the Pfaffian of the following matrix:

$$M = \begin{bmatrix} 0 & v & w & x & y & z \\ -v & 0 & 0 & z & -x & 0 \\ -w & 0 & 0 & 0 & v & -y \\ -x & -z & 0 & 0 & 0 & w \\ -y & x & -v & 0 & 0 & 0 \\ -z & 0 & y & -w & 0 & 0 \end{bmatrix}.$$

Its quadratic Pfaffians are given by

$$c_{ij} = (-1)^{i+j+1}(a_{pq}a_{rs} - a_{pr}a_{qs} + a_{ps}a_{qr}),$$

where $p < q < r < s$, $(pqrsij)$ is a permutation of (123456), and $(-1)^{i+j+1}$ is nothing but its sign. A direct computation shows that the 15 quadratic Pfaffians are linearly independent. $\square$

This ends the proof of Theorem 2.2.

3. Minimal sections of 2-dimensional conic bundle

Let X be a generic cubic threefold. To prove the irreducibility of the fibers of the Abel-Jacobi map Φ of Theorem 2.1, we will use other Abel–Jacobi maps. Let us fix a line l_0 in X, and denote by $\Phi_{d,g}$ the Abel–Jacobi map of the family $H_{d,g}$ of curves of degree d and of arithmetic genus g in X having dl_0 as reference curve. The precise domain of definition of $\Phi_{d,g}$ will be specified in the context in each particular case. So, $\Phi_{5,1}$ will be exactly the above map Φ defined on $\mathcal{H}$.

We will provide a description of $\Phi_{4,0}$, obtained by an application of the techiniques of [I]. This map is defined on the family of normal rational quartics in X. For completeness, we will mention a similar description of $\Phi_{3,0}$, the Abel–Jacobi map of twisted rational cubics in X. As was proved in [MT], these families of curves are irreducible for a generic X.

Let $L_0 \subset X$ be a generic line, $p : \tilde{X} \longrightarrow \mathbb{P}^2$ the projection from L_0, giving to $\tilde{X} = \mathrm{Blowup}_{L_0}(X)$ a structure of a conic bundle. Let $C \subset \mathbb{P}^2$ be a generic conic, then $Y = p^{-1}(C)$ is a 2-dimensional conic bundle, and $p_Y = p|_Y : Y \longrightarrow C$ is the conic bundle structure map. It is well known (see [B]), that the discriminant curve $\Delta \subset \mathbb{P}^2$ of p is a smooth quintic, and the components of the reducible conics $\mathbb{P}^1 \vee \mathbb{P}^1$ over points of Δ are parametrized by a non-ramified two-sheeted covering $\pi : \tilde{\Delta} \longrightarrow \Delta$. As C is generic, there are 10 distinct points in $\Delta \cap C$, giving us 10 pairs of lines $\{l_1 \cup l_1' \cup \ldots \cup l_{10} \cup l_{10}'\} = p^{-1}(\Delta \cap C)$. We will identify the components l of reducible fibers of p with points of $\tilde{\Delta}$, so that $\{l_1, l_1', \ldots, l_{10}, l_{10}'\} = \pi^{-1}(\Delta \cap C) \subset \tilde{\Delta}$. Let $p_\alpha : Y_\alpha \longrightarrow C$ be any of

the 2^{10} ruled surfaces obtained by contracting the l'_i with $i \in \alpha$ and the l_j with $j \notin \alpha$, where α runs over the subsets of $\{1, 2, \dots, 10\}$. Then the Y_α are divided into two classes: even and odd surfaces, according to the parity of the integer $n \geq 0$ such that $Y_\alpha \simeq \mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$. Remark, that the surfaces Y_α are in a natural one-to-one correspondence with effective divisors D of degree 10 on $\tilde{\Delta}$ such that $\pi_* D = \Delta \cap C$. The 10 points of such a divisor correspond to lines (l_i or l'_i) which are not contracted by the map $Y \longrightarrow Y_\alpha$. For a surface Y_α, associated to an effective divisor D of degree 10, we will use the alternative notation Y_D.

The next theorem is a particular case of the result of [I].

Theorem 3.1. *Let X be a generic cubic threefold, $C \in \mathbb{P}^2$ a generic conic. Then, in the above notation, the following assertions hold:*

(i) There are only two isomorphism classes of surfaces among the Y_α: $Y_{odd} \simeq \mathbb{F}_1$ and $Y_{even} \simeq \mathbb{F}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

(ii) The family $\mathcal{C}_-$ of the proper transforms in X of (-1)-curves in each one of the odd surfaces $Y_\alpha \simeq \mathbb{F}_1$ over all sufficiently generic conics $C \subset \mathbb{P}^2$ is identified with a dense open subset in the family of twisted rational cubic curves $C^3 \in X$ meeting L_0 at one point.

(iii) Let $\Phi_{3,0}$ be the Abel–Jacobi map of the family of rational twisted cubics. Let $\Phi_- = \Phi_3|_{\mathcal{C}_-}$ be its restriction. Then Φ_- is onto an open subset of the theta divisor of $J^2(X)$. For generic $C^3 \in \mathcal{C}_-$, which is a proper transform of the (-1)-curve in the ruled surface Y_α associated to an effective divisor D_α of degree 10 on $\tilde{\Delta}$, the fiber $\Phi_-^{-1}\Phi_-(C^3)$ can be identified with an open subset of $\mathbb{P}^1 = |D_\alpha|$ by the following rule:

$$D \in |D_\alpha| \ \mapsto \ \left| \begin{array}{l} \textit{the proper transform in } X \textit{ of the } (-1)\textit{-curve in} \\ Y_D \textit{ if } Y_D \simeq \mathbb{F}_1 \end{array} \right.$$

(iv) Let $\mathcal{C}_+$ be the family of the proper transforms in X of the curves in the second ruling on any one of the even surfaces $Y_\alpha \simeq \mathbb{P}^1 \times \mathbb{P}^1$ for all sufficiently generic conics C; the second ruling means the one which is different from that consisting of fibers of π_α. Then $\mathcal{C}_+$ is identified with a dense open subset in the family of normal rational quartic curves $C^4 \in X$ meeting L_0 at two points.

(v) Let $\Phi_{4,0}$ be the Abel–Jacobi map of the family of rational normal quartics. Let $\Phi_+ = \Phi_{4,0}|_{\mathcal{C}_+}$ be its restriction. Then Φ_+ is onto an open subset of $J^2(X)$. For generic $C^4 \in \mathcal{C}_+$ which is the proper transform of a curve on the ruled surface Y_α associated to an effective divisor D_α of degree 10 on $\tilde{\Delta}$, where $\dim |D_\alpha| = 0$ and the fiber $\Phi_+^{-1}\Phi_+(C^4) \simeq \mathbb{P}^1$ consists of the proper transforms of all the curves of the second ruling on Y_α.

The irreducibility of $\Phi_+^{-1}\Phi_+(C^4)$ in the above statement is an essential ingredient of the proof of the following theorem, which is the main result of the paper.

Theorem 3.2. *Let X be a nonsingular cubic threefold. Then the degree of the étale map Ψ from Theorem 2.1 is 1. Equivalently, all the fibers of the Abel–Jacobi map Φ are isomorphic to $\mathbb{P}^5$.*

This obviously implies:

Corollary 3.3. *The open set $M \subset M_X(2;0,2)$ in the moduli space of vector bundles on X parametrizing those stable rank 2 vector bundles which arise via Serre's construction from normal elliptic quintics is isomorphic to an open subset in the intermediate Jacobian of X.*

We will start by the following lemma.

Lemma 3.4. *Let X be a generic cubic threefold. Let $z \in J^2(X)$ be a generic point, $\mathcal{H}_i(z) \simeq \mathbf{P}^5$ any component of $\Phi^{-1}(z)$. Then, for any line $l \subset X_3$, the family*

$\mathcal{H}_{l;i}(z) := \{C \in \mathcal{H}_i(z) : C = l + C'$, where C' is a curve of degree 4 $\}$ is isomorphic to $\mathbf{P}^1$.

Proof. By Theorem 2.1, the curve C represented by the generic point of $\mathcal{H}_i(z)$ is a (smooth) normal elliptic quintic. Let $\mathcal{E} = \mathcal{E}_C$ be the associated vector bundle, represented by the point $\phi([C]) \in M$. Choose any representation of X as a linear section of the Pfaffian cubic Ξ' as in Theorem 2.2, so that $\mathcal{E}(1) \simeq \mathcal{K}^\vee$. The projective space $\mathcal{H}_i(z)$ is naturally identified with $\mathbb{P}^{5\vee} = \mathbb{P}(E^\vee)$. This follows from the proof of Theorem 2.2. Indeed, the curves C represented by points of $\mathcal{H}_i(z)$ are exactly the zero loci of the sections of $\mathcal{E}(1)$, and the latter are induced by linear forms on E via the natural surjection $E_X \longrightarrow \mathcal{K}^\vee$. The zero loci of these sections are of the form $\mathrm{Cl}^{-1}(\sigma_{11}(L))$, where $L \in \mathbb{P}^{5\vee}$ runs over all the hyperplanes in $\mathbb{P}^5$.

Let l be a line in X. By Lemma 2.4, the quadratic pencil of lines with base $\mathrm{Cl}(l)$ sweeps out a quadric surface $Q(l)$ of rank 3 or 4. Let $<Q(l)>\simeq \mathbb{P}^3$ be the linear span of $Q(l)$ in $\mathbb{P}^5$. Then l is a component of $\mathrm{Cl}^{-1}(\sigma_{11}(L))$ if and only if $<Q(l)>\subset L$. Such hyperplanes L form the pencil $<Q(l)>^\vee\simeq \mathbb{P}^1$ in $\mathbb{P}^{5\vee}$. Obviously, the pencil $\{\mathrm{Cl}^{-1}(\sigma_{11}(L)) \mid L \in <Q(l)>^\vee\}$ contains exactly all the curves, represented by points of $\mathcal{H}_i(z)$ and having l as an irreducible component. $\square$

Now our aim is to show that the generic member of $\mathcal{H}_{l;i}(z)$ is a rational normal quartic having l as one of its chords. Then we will be able to apply the description of such curves given by Theorem 3.1, (iv), (v).

4. Smoothing $C' + l$

Let X be a nonsingular cubic threefold, $C = C' + l \subset X$ a rational normal quartic plus one of its chords. Then one can apply Serre's construction (3) to C to obtain a self-dual rank 2 vector bundle $\mathcal{E} = \mathcal{E}_C$ in $M_X(2;0,2)$ like it was done in [MT] for a nonsingular C. One proves directly that $\mathcal{E}$ possesses all the essential properties of the vector bundles constructed from normal elliptic quintics. First of all, our C is a locally complete intersection in X with trivial canonical sheaf ω_C, and this implies (see the proofs of Lemma 2.1 and Corollary 2.2 in loc. cit.) that $\mathrm{Ext}^1(\mathcal{I}_C(2), \mathcal{O}_X) \simeq H^0(C, \omega_C) \simeq \mathbb{C}$ and that $\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{I}_C(2), \mathcal{O}_X) = \mathcal{E}xt^2_{\mathcal{O}_X}(\mathcal{O}_C, \omega_X) = \omega_C$, so that $\mathcal{E}$ is uniquely determined up to isomorphism and is locally free. One can also easily show that $h^0(\mathcal{I}_C(1)) = h^1(\mathcal{I}_C(1)) = h^2(\mathcal{I}_C(1)) = 0$, and this implies (see the proofs of Corollary 2.4, Proposition 2.6 and Lemma 2.8 in loc. cit.) the stability of $\mathcal{E}$ and the fact that the zero loci of nonproportional sections of $\mathcal{E}(1)$ are distinct complete intersection linearly normal quintic curves. Further, remark that $h^0(\mathcal{I}_C(2)) = 5$ (the basis of $H^0(\mathcal{I}_C(2))$ is given in appropriate coordinates in (9) below); the restriction exact sequence

$$0 \longrightarrow \mathcal{I}_C(k) \longrightarrow \mathcal{O}_X(k) \longrightarrow \mathcal{O}_C(k) \longrightarrow 0 \tag{4}$$

with $k = 2$ implies also $h^i(\mathcal{I}_C(2)) = 0$ for $i > 0$. One deduces from here $h^0(\mathcal{E}(1)) = 6, h^i(\mathcal{E}(1)) = 0$ for $i > 0$. Hence the sections of $\mathcal{E}$ define a $\mathbb{P}^5$ in Hilb^{5n}_X.

We want to show that, generically, this $\mathbb{P}^5$ is of the form $\mathcal{H}_i(z)$, that is $\mathcal{E}$ has a section whose zero locus is a (smooth) normal elliptic quintic. First, we make a routine verification that C can be smoothed into a normal elliptic quintic, that is $[C] \in \mathrm{Hilb}^{5n}_X$ is in the closure of $\mathcal{H}^*$. Afterwards, we will show that the smoothing can be effectuated inside the $\mathbb{P}^5$ of zero loci of sections of $\mathcal{E}$. To this end, we will find an *example* of $(X, \mathcal{E})$, in which the locus of the curves of type 'normal rational quartic plus its chord' inside the $\mathbb{P}^5$ has at least one 3-dimensional component. By a standard dimension count, this will imply that all the components of this locus are 3-dimensional for generic $(X, \mathcal{E})$, and that the generic point of the $\mathbb{P}^5$ represents a nonsingular curve.

Lemma 4.1. *Let X be a nonsingular cubic threefold, $C = C' + l \subset X$ a rational normal quartic plus one of its chords, $\mathcal{E}$ the vector bundle defined in 4. Then the following assertions are true:*

(i) $h^i(\mathcal{E}(-1)) = 0 \ \ \forall\ i \in \mathbb{Z}$, hence $\mathcal{E}$ is an instanton vector bundle of charge 2. Further, $h^0(\mathcal{E} \otimes \mathcal{E}) = 1$, $h^1(\mathcal{E} \otimes \mathcal{E}) = 5, h^2(\mathcal{E} \otimes \mathcal{E}) = h^3(\mathcal{E} \otimes \mathcal{E}) = 0$, hence $M_X(2;0,2)$ is smooth of dimension 5 at $[\mathcal{E}]$.

ii) $h^0(\mathcal{N}_{C/X}) = 10, h^1(\mathcal{N}_{C/X}) = 0$*, hence* Hilb^{5n}_X *is smooth of dimension 10 in* $[C]$*. Moreover, if we assume that* $\mathcal{N}_{C'/X} \not\simeq \mathcal{O} \oplus \mathcal{O}(6)$ *or* $\mathcal{N}_{l/X} \not\simeq \mathcal{O}(-1) \oplus \mathcal{O}(1)$*, then* C *is strongly smoothable and a sufficiently small deformation* $\mathfrak{C} \longrightarrow U$ *of* C *parametrizes curves of only the following three types: (a) for* u *in a dense open subset of* U*,* C_u *is a normal elliptic quintic; (b) over on open subset of a divisor* $\Delta_1 \subset U$*,* C_u *is a linearly normal rational curve with only one node as singularity; (c) over a closed subvariety of pure codimension 2* $\Delta_2 \subset U$*,* C_u *is of the same type as* C*, that is a normal rational quartic plus one of its chords.*

Proof. As concerns the numerical values for the h^i, the proof goes exactly as that of Lemma 2.7 in [MT] with only one modification: the authors used there the property of a normal elliptic quintic $h^0(\mathcal{N}_{C/\mathbb{P}^4}(-2)) = 0$, proved in Proposition V.2.1 of [Hu]. Here we should verify directly this property for our curve $C = C' + l$. This is an easy exercise: one can use the identifications of the normal bundles of C', l

$$\mathcal{N}_{C'/\mathbb{P}^4} \simeq 3\mathcal{O}_{\mathbb{P}^1}(6)\ , \qquad \mathcal{N}_{l/\mathbb{P}^4} \simeq 3\mathcal{O}_{\mathbb{P}^1}(1) \tag{5}$$

and the three natural exact sequences

$$0 \longrightarrow \mathcal{N}_{C/W} \longrightarrow \mathcal{N}_{C/W}|_{C'} \oplus \mathcal{N}_{C/W}|_{l} \longrightarrow \mathcal{N}_{C/W} \otimes \mathbb{C}_S \longrightarrow 0, \tag{6}$$

$$0 \longrightarrow \mathcal{N}_{C'/W} \longrightarrow \mathcal{N}_{C/W}|_{C'} \longrightarrow T^1_S \longrightarrow 0, \tag{7}$$

$$0 \longrightarrow \mathcal{N}_{l/W} \longrightarrow \mathcal{N}_{C/W}|_{l} \longrightarrow T^1_S \longrightarrow 0, \tag{8}$$

where $S = \{P_1, P_2\} = C' \cap l$, $\mathbb{C}_S = \mathbb{C}_{P_1} \oplus \mathbb{C}_{P_2}$ is the sky-scraper sheaf with the only nonzero stalks at P_1, P_2 equal to $\mathbb{C}$, $W = \mathbb{P}^4$, and T^1_S denotes Schlesinger's T^1 of a singularity; we have $T^1_S \simeq \mathbb{C}_S$ for nodal curves.

The values of $h^i(\mathcal{E} \otimes \mathcal{E})$ in (i) imply the stated properties of the moduli space, because $\mathcal{E}$ is self-dual, and so $h^i(\mathcal{E} \otimes \mathcal{E}) = \dim \mathrm{Ext}^i(\mathcal{E}, \mathcal{E})$.

For the remaining assertions of (ii), we will apply Theorem 4.1 of [HH] [1]. It states that if the elementary transformations of the normal bundles to C', l satisfy $H^1(C', \mathrm{elm}^+_{P_i} \mathcal{N}_{l/X}) = H^1(l, \mathrm{elm}^-_S \mathcal{N}_{C'/X}) = 0$, then C is strongly smoothable. In fact, by (5) and

$$0 \longrightarrow \mathcal{N}_{C'/X} \longrightarrow \mathcal{N}_{C/\mathbb{P}^4} \longrightarrow \mathcal{N}_{X/\mathbb{P}^4}|_C \longrightarrow 0\ ,$$

we see that $\mathcal{N}_{C'/X} \simeq \mathcal{O}(a) \oplus \mathcal{O}(b)$ with $a + b = 6, 0 \le a \le b \le 6$. By [CG], $\mathcal{N}_{l/X} \simeq 2\mathcal{O}$ or $\mathcal{O}(-1) \oplus \mathcal{O}(1)$, so $\mathrm{elm}^+_{P_i} \mathcal{N}_{l/X} = \mathcal{O}(-1) \oplus$

[1] Hartshorne–Hirschowitz formulated it for nodal curves in $\mathbb{P}^3$, but the proof and the techniques of the paper remain valid if one replaces $\mathbb{P}^3$ by any nonsingular projective variety; see Remark 4.1.1 in [HH].

$\mathcal{O}(2)$ or $\mathcal{O} \oplus \mathcal{O}(1)$. For C', $\operatorname{elm}_S^- \mathcal{N}_{C'/X}$ may be one of the sheaves $\mathcal{O}(a-2) \oplus \mathcal{O}(b)$, $\mathcal{O}(a-1) \oplus \mathcal{O}(b-1)$, or $\mathcal{O}(a) \oplus \mathcal{O}(b-2)$. So, the hypotheses of the theorem may be not verified only if $a = 0, b = 6$. In interchanging the roles of C', l and assuming that $\mathcal{N}_{l/X} \simeq 2\mathcal{O}$, we can see that $\operatorname{elm}_S^- \mathcal{N}_{l/X} \simeq 2\mathcal{O}(-1)$ or $\mathcal{O} \oplus \mathcal{O}(-2)$. The second case is impossible, because l and the tangent directions of C' at the points P_i are not coplanar, so the centers of the elementary transformation $\tilde{P}_i \in \mathbb{P}(\mathcal{N}_{l/X}|_{P_i}) \simeq \mathbb{P}^1$, corresponding to the directions of C' at P_i, do not lie on the same section of $\mathbb{P}(\mathcal{N}_{l/X}) = \mathbb{P}^1 \times \mathbb{P}^1$.

Thus, in both cases the theorem can be applied, and we conclude that the natural maps $\delta_i : H^0(\mathcal{N}_{C/X}) \longrightarrow T^1_{P_i} C = \mathbb{C}_{P_i}$ are surjective. Hence the discriminant divisor $\Delta_1 \subset U$ has locally analytically two nonsingular branches with tangent spaces $\ker \delta_i \subset H^0(\mathcal{N}_{C/X}) = T_{[C]} \mathrm{Hilb}^{5n}_X$, each unfolding only one of the two singular points of C, and their transversal intersection Δ_2 parametrizes the deformations preserving the two singular points. To conclude the proof, remark that the linear normality and $p_a(C)$ are preserved under small deformations. □

Lemma 4.2. *Let X be a generic cubic threefold, $C = C' + l \subset X$ a generic rational normal quartic plus one of its chords, $\mathcal{E}$ the vector bundle defined above. Let $\mathcal{H}_{\mathcal{E}} \subset \mathrm{Hilb}^{5n}_X$ be the $\mathbb{P}^5$ of zero loci of sections of $\mathcal{E}(1)$. Then the assumption of Lemma 4.1, (ii) for the normal bundle of l is verified and, moreover, $\dim \Delta_i \cap \mathcal{H}_{\mathcal{E}} = 5 - i$ for $i = 1, 2$. This implies that $C' + l$ can be smoothed not only inside Hilb^{5n}_X, but also inside $\mathcal{H}_{\mathcal{E}}$.*

Proof. We have to show that $\mathcal{N}_{l/X} \simeq 2\mathcal{O}$ and that the natural map $T_{[C]} \mathrm{Hilb}^{5n}_X \longrightarrow T^1_S C$ remains surjective when restricted to $T_{[C]} \mathcal{H}_{\mathcal{E}} \subset T_{[C]} \mathrm{Hilb}^{5n}_X$. It suffices to do this only for one special cubic threefold X and for one special C, because both conditions are open. So, choose a curve C of type $C' + l$ in $\mathbb{P}^4$, then a cubic X passing through C. Take, for example, the closures of the following affine curves:

$$C' = \{x_1 = t,\ x_2 = t^2,\ x_3 = t^3,\ x_4 = t^4\}\ , \quad l = \{x_1 = x_2 = x_3 = 0\}\ .$$

The family of quadrics passing through C is 5-dimensional with generators

$$Q_1 = x_2 - x_1^2,\ Q_2 = x_3 - x_1 x_2,\ Q_3 = x_1 x_3 - x_2^2,$$
$$Q_4 = x_1 x_4 - x_2 x_3,\ Q_5 = x_2 x_4 - x_3^2\ . \quad (9)$$

The cubic hypersurface in $\mathbb{P}^4$ with equation $\sum \alpha_i(x) Q_i$ is nonsingular for generic linear forms $\alpha_i(x)$, so we can choose X to be of this form. We verified, in using the Macaulay program [BS], that the choice $\alpha_1 =$

$0, \alpha_2 = -1, \alpha_3 = x_2, \alpha_4 = -x_1, \alpha_5 = x_4$ yields a nonsingular $X = \{x_1x_2 - x_2^3 - x_3 + 2x_1x_2x_3 - x_1^2x_4 - x_3^2x_4 + x_2x_4^2 = 0\}$ such that $\mathcal{N}_{l/X} \simeq 2\mathcal{O}$.

Look at the following commutative diagram with exact rows and columns, where the first row is the restriction of (3) to the subsheaf of the sections of $\mathcal{E}(1)$ vanishing along C.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[d, equal] & \mathcal{E}(1) \otimes \mathcal{I}_C \arrow[r] \arrow[d] & \mathcal{I}_C^2(2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{E}(1) \arrow[r] \arrow[d] & \mathcal{I}_C(2) \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{N}_{C/X} \arrow[r, equal] \arrow[d] & \mathcal{N}^\vee_{C/X}(2) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \qquad (10)$$

It allows to identify the tangent space $T_{[\mathcal{E}]}\mathcal{H}_\mathcal{E} = H^0(\mathcal{E}(1))/H^0(\mathcal{E}(1)\otimes \mathcal{I}_C)$ with the image of $H^0(\mathcal{I}_C(2))$ in $H^0(\mathcal{N}_{C/X}) = H^0(\mathcal{I}_C(2)/\mathcal{I}_C(2)^2)$. So, we have to show that the derivative $d : H^0(\mathcal{I}_C(2)) \longrightarrow T^1_S C$ is surjective. Using the basis (9) of $H^0(\mathcal{I}_C(2))$, we easily verify that this is the case (in fact, dQ_1, dQ_2 generate $T^1_S C$). $\square$

Lemma 4.3. *With the hypotheses of Lemma 3.4, the family $\mathcal{C}_i(z)$ of curves of the form $C' + l$ in $\mathcal{H}_i(z)$, where C' is a rational normal quartic and l one of its chords, is non-empty and equidimensional of dimension* 3.

Proof. According to [MT], the family of rational normal quartics in a nonsingular cubic threefold X has dimension 8, and is irreducible for generic X. By Theorem 1.2, each rational normal quartic C' has exactly 16 chords l in X, so the family $\Delta_2 = \Delta_2(X)$ of pairs $C' + l$ is equidimensional of dimension 8. It suffices to verify that one of the components of Δ_2, say $\Delta_{2,0}$, meets $\mathcal{H}_i(z)$ at some point b with local dimension $\dim_b \Delta_{2,0} \cap \mathcal{H}_i(z) = 3$ for one special cubic threefold X, for one special z and for at least one i. But this was done in the previous lemma. Indeed, the fact that C can be smoothed inside $\mathcal{H}_\mathcal{E}$ implies that $\mathcal{E} \in \mathcal{H}$, hence $\mathcal{H}_\mathcal{E} = \mathcal{H}_i(z)$ for some i, z. The assertion for general X, z follows by the relativization over the family of cubic threefolds and the standard count of dimensions. $\square$

Corollary 4.4. *With the hypotheses of Lemma 3.4, let l be a generic line in X. Then the generic member of the pencil $\mathcal{H}_{l;i}(z)$ is a rational normal quartic plus one of its chords.*

Proof. We know already that the family of pairs $C' + l \in \mathcal{C}_i(z)$ is 3-dimensional. Now we are to show that the second components l of these pairs move in a dense open subset in the Fano surface F of X. This can be done by an infinitesimal argument: let $C = C' + l$ be such a pair. Then a small neighborhood of $[C]$ in $\mathcal{H}_i(z)$ contains a smooth subvariety D of codimension 2 with tangent space $T_{[C]}D = \ker\{T_{[C]}\mathcal{H}_i(z) \longrightarrow T^1_S C\}$, which parametrizes the curves $C'' + l$ of the same type. It suffices to show that the natural projection of $T_{[C]}D$ to $T_{[l]}F = H^0(\mathcal{N}_{l/X})$ is surjective.

The exact triples (6), (7), (8) with $W = X$ together with the observation that all the H^1's vanish imply that the natural map $H^0(\mathcal{N}_{C/X}) \longrightarrow H^0(\mathcal{N}_{C/X}|_l)$ is surjective and that it restricts to a surjective map between the kernels of the respective maps to $T^1_S C$:

$$T_{[C]}\Delta_2 X = \ker\{H^0(\mathcal{N}_{C/X}) \longrightarrow T^1_S C\} \twoheadrightarrow H^0(\mathcal{N}_{l/X}) = \ker\{H^0(\mathcal{N}_{C/X}|_l) \longrightarrow T^1_S C\}.$$

We want to see that it will remain surjective even if we shrink its source to $T_C D \subset T_{[C]}\Delta_2 X$.

Let $\mathcal{E} = \mathcal{E}_C$ be as above. Then the triple (3) determines an isomorphism $\mathcal{N}_{C/X} = \mathcal{E}(1)|_C$ in such a way that $T_{[C]}\mathcal{H}_i(z)$ is the image of the restriction map $H^0(\mathcal{E}(1)) \longrightarrow H^0(\mathcal{E}(1)|_C)$. Hence we can identify $\mathcal{N}_{C/X}|_l$ with $\mathcal{E}(1)|_l$. Look at the exact triple

$$0 \longrightarrow \mathcal{E}(1) \otimes \mathcal{I}_l \longrightarrow \mathcal{E}(1) \longrightarrow \mathcal{E}(1)|_l \longrightarrow 0.$$

By Lemma 3.4, $h^0(\mathcal{E}(1) \otimes \mathcal{I}_l) = 2$. By Lemma 4.2, we have $\mathcal{N}_{l/X} \simeq 2\mathcal{O}$ for generic $C' + l$. As in the proof of Lemma 4.1, the fact that the centers of the elementary transformation $\mathrm{elm}^+_S \mathcal{N}_{l/X} \simeq \mathcal{N}_{C/X}|_l$ of $\mathcal{N}_{l/X}$ do not lie on the same section of $\mathbb{P}(\mathcal{N}_{l/X}) = \mathbb{P}^1 \times \mathbb{P}^1$ implies that $\mathcal{N}_{C/X}|_l \simeq 2\mathcal{O}(1)$ and that the map $\mathcal{N}_{C/X}|_l \longrightarrow T^1_S C$ is surjective. As $h^0(\mathcal{E}(1)) = 6$, the last exact triple gives the surjectivity of $H^0(\mathcal{E}(1)) \longrightarrow H^0(\mathcal{N}_{C/X}|_l)$. Restricting the map to the kernels of the natural surjections onto $T^1_S C$, we obtain the result.

□

5. Fibers of $\Phi_{4,0}, \Phi_{5,1}$ and periods of varieties V_{14}

Now we are able to prove Theorem 3.2. Let X be a generic cubic threefold. Let $\Phi_{1,0}$, $\Phi_{4,0}$, resp. $\Phi^* = \Phi^*_{5,1}$ be the Abel–Jacobi map of lines, rational normal quartics, resp. elliptic normal quintics. We

will use the notation Φ, or $\Phi_{5,1}$ for the extension of Φ^* defined in the statement of Theorem 2.1. By Lemma 4.2, the generic curves of the form $C'+l$, where C' is a rational normal quartic and l one of its chords, are elements of $\mathcal{H}$, the domain of Φ.

Proof of Theorem 3.2. Let $z \in J^2(X)$ be a generic point, $\mathcal{H}_i(z) \simeq \mathbf{P}^5$ any component of $\Phi^{-1}(z)$. Choose a generic line l on X. In the notations of Lemma 3.4, the number of pencils $\mathcal{H}_{l;i}(z) \simeq \mathbb{P}^1$ with generic member $C_i'+l$, where C_i' is a rational normal quartic meeting l quasi-transversely at 2 points, and mapped to the same point z of the intermediate Jacobian, is equal to the degree d of Ψ. Now look at the images of the curves C_i' arising in these pencils under the Abel–Jacobi map $\Phi_{4,0}$. Denoting AJ the Abel–Jacobi map on the algebraic 1-cycles homologous to 0, we have $AJ((C_i'+l)-(C_j'+l)) = AJ(C_i'-C_j') = z-z=0$. Hence $\Phi_{4,0}(C_i') = \Phi_{4,0}(C_j')$ is a constant point $z' \in J^2(X)$. According to Theorem 3.1, the family of the normal rational quartics in a generic fiber of $\Phi_{4,0}$ meeting a generic line at two points is irreducible and is parametrized by (an open subset of) a $\mathbb{P}^1$. The point z' is a generic one, because $\Phi_{4,0}$ is dominant, and every rational normal quartic has at least one chord. Hence $d=1$ and we are done. $\square$

Corollary 5.1. *$M,\mathcal{H}$ are irreducible and the degree of Ψ is 1 not only for a generic cubic X, but also for every nonsingular one.*

Proof. One can easily relativize the constructions of $\mathcal{H}, M, \Phi, \phi, \Psi$, etc. over a small analytic (or étale) connected open set U in the parameter space $\mathbb{P}^{34}$ of 3-dimensional cubics, over which all the cubics X_u are nonsingular. We have to restrict ourselves to a "small" open set, because we need a local section of the family $\{\mathcal{H}_u\}$ in order to define the maps Φ, Ψ.

The fibers $\mathcal{H}_u, M_u$ are equidimensional and nonsingular of dimensions $10, 5$ respectively. Moreover, it is easy to see that a normal elliptic quintic C_0 in a special fiber X_{u_0} can be deformed to the neighboring fibers X_u. Indeed, one can embed the pencil $\lambda X_{u_0} + \mu X_u$ into the linear system of hyperplane sections of a 4-dimensional cubic Y and show that the local dimension of the Hilbert scheme of Y at $[C_0]$ is 15, which implies that C_0 deforms to all the nearby and hence to all the nonsingular hypeplane sections of Y.

Hence the families $\{\mathcal{H}_u\}, \{M_u\}$ are irreducible, flat of relative dimensions 10, resp. 5 over U, and the degree of Ψ is constant over U. If there is a reducible fiber M_u, then the degree sums up over its irreducible components, so it has to be strictly greater than 1. But we know, that d is 1 over the generic fiber, hence all the fibers are irreducible and $d=1$ for all u. $\square$

We are going to relate the Abel–Jacobi mapping of elliptic normal quintics with that of rational normal quartics. With our convention for the choice of reference curves in the form dl_0 for a line l_0, fixed once and forever, we have the identity

$$\Phi_{5,1}(C'+l) = \Phi_{4,0}(C') + \Phi_{1,0}(l) \ .$$

Theorem 5.2. *Let X be a generic cubic threefold, $z \in J^2(X)$ a generic point. Then the corresponding fiber $\Phi_{4,0}^{-1}(z)$ is an irreducible nonsingular variety of dimension 3, birationally equivalent to X.*

Proof. As we have already mentioned in the proof of Lemma 4.3, the family $H_{4,0}$ of rational normal quartics in X is irreducible of dimension 8. The nonsingularity of $H_{4,0}$ follows from the evaluation of the normal bundle of a rational normal quartic in the proof of Lemma 4.1. We saw also that $\Phi_{4,0} : H_{4,0} \longrightarrow J^2(X)$ is dominant, so the generic fiber is equidimensional of dimension 3 and we have to prove its irreducibility.

Let $\pi : \tilde{U} \longrightarrow U$ be the quasi-finite covering of $U = \Phi(\mathcal{H})$ parametrizing the irreducible components of the fibers of $\Phi_{4,0}$ over points of U. Let $z \in U$ be generic, and $\mathcal{H}_z \simeq \mathbb{P}^5$ the fiber of Φ. By Corollary 4.4, for a generic line l, we can represent z as $\Phi_{4,0}(C') + \Phi_{1,0}(l)$ for a rational normal quartic C' having l as one of its chords. Let $\kappa : U \dashrightarrow \tilde{U}$ be the rational map sending z to the component of $\Phi_{4,0}^{-1}\Phi_{4,0}(C')$ containing C'. Let $\lambda = \pi \circ \kappa$. Theorem 3.1 implies that λ is dominant. Hence it is generically finite. Then κ is also generically finite, and we have for their degrees $\deg \lambda = (\deg \pi)(\deg \kappa)$.

Let us show that $\deg \lambda = 1$. Let z, z' be two distinct points in a generic fiber of λ. By Theorem 3.1, $\Phi_{4,0}^{-1}\Phi_{4,0}(C')$ contains only one pencil of curves of type $C'' + l$, where l is a fixed chord of C', and C'' is a rational normal quartic meeting l in 2 points. But Lemma 3.4 and Corollary 4.4 imply that both $\mathcal{H}_z$ and $\mathcal{H}_{z'}$ contain such a pencil. This is a contradiction. Hence $\deg \lambda = \deg \pi = \deg \kappa = 1$.

Now, choose a generic rational normal quartic C' in X. We are going to show that $\Phi_{4,0}^{-1}\Phi_{4,0}(C')$ is birational to some V_{14}, associated to X, and hence birational to X itself. Namely, take the V_{14} obtained by the Tregub–Takeuchi transformation χ from X with center C'. Let $x \in V_{14}$ be the indeterminacy point of χ^{-1}. The pair (x, V_{14}) is determined by (C', X) uniquely up to isomorphism, because V_{14} is the image of X under the map defined by the linear system $|\mathcal{O}_X(8) - 5C'|$ and x is the image of the unique divisor of the linear system $|\mathcal{O}_X(3 - 2C')|$.

By Theorem 1.2, (iii), a generic $\xi \in V_{14}$ defines an inverse map of Tregub–Takeuchi type from V_{14} to the same cubic X. As X is generic, it has no biregular automorphisms, and hence this map defines

a rational normal quartic Γ in X. We obtain the rational map $\alpha : V_{14} \dashrightarrow H_{4,0}$, $\xi \mapsto [\Gamma]$, whose image contains $[C']$. As $h^{1,0}(V_{14}) = 0$, the whole image $\alpha(V_{14})$ is contracted to a point by the Abel–Jacobi map. Hence, to show that $\Phi_{4,0}^{-1}\Phi_{4,0}(C')$ is birationally equivalent to V_{14}, it suffices to see that α is generically injective. This follows from the following two facts: first, the pair (ξ, V_{14}) is determined by (Γ, X) uniqueley up to isomorphism, and second, a generic V_{14} has no biregular automorphisms. If there were two points $\xi, \xi' \in V_{14}$ giving the same Γ, then there would exist an automorphism of V_{14} sending ξ to ξ', and hence $\xi = \xi'$. Another proof of the generic injectivity of α is given in Proposition 5.6.

We did not find an appropriate reference for the second fact, so we prove it in the next lemma. □

Lemma 5.3. *A generic variety V_{14} has no nontrivial biregular automorphisms.*

Proof. As V_{14} is embedded in $\mathbb{P}^9$ by the anticanonical system, any biregular automorphism g of V_{14} is induced by a linear automorphism of $\mathbb{P}^9$. Hence it sends conics to conics, and thus defines an automorphism $F(g) : F(V_{14}) \longrightarrow F(V_{14})$ of the Fano surface $F(V_{14})$, parametrizing conics on V_{14}. In [BD], the authors prove that the Hilbert scheme $\mathrm{Hilb}^2(S) = S^{[2]}$ parametrizing pairs of points on the K3 surface S of degree 14 in $\mathbb{P}^8$ is isomorphic to the Fano 4-fold $F(V_3^4)$ parametrizing lines on V_3^4, where V_3^4 is the 4-dimensional linear section of the Pfaffian cubic in $\mathbb{P}^{14}$ associated to S. The same argument shows that $F(V_{14}) \simeq F(X)$, where X is the cubic 3-fold associated to V_{14}, and $F(X)$ is the Fano surface parametrizing lines on X.

Hence g induces an automorphism f of $F(X)$. Let f^* be the induced linear automorphism of $\mathrm{Alb}(F(X)) = J^2(X)$, and $T_0 f^*$ its differential at the origin. By [Tyu], the projectivized tangent cone of the theta divisor of $J^2(X)$ at 0 is isomorphic to X, so $T_0 f^*$ induces an automorphism of X. V_{14} being generic, X is also generic, so $\mathrm{Aut}(X) = \{1\}$. Hence $f^* = \mathrm{id}$. By the Tangent Theorem for $F(X)$ [CG], $\Omega^1_{F(X)}$ is identified with the restriction of the universal rank 2 quotient bundle $\mathcal{Q}$ on $G(2,5)$, and all the global sections of $\Omega^1_{F(X)}$ are induced by linear forms L on $\mathbb{P}^4$ via the natural map $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1)) \otimes \mathcal{O}_{G(2,5)} \twoheadrightarrow \mathcal{Q}$. Hence the fact that f acts trivially on $H^0(\Omega^1_{F(X)}) = T_0^* J^2(X)$ implies that f permutes the lines $l \in \{L = 0\} \cap X$ lying in one hyperplane section of X. For general L, there are 27 lines l, and in taking two hypeplane sections $\{L_1 = 0\}, \{L_2 = 0\}$ which have only one common line, we conclude that f fixes the generic point of $F(X)$. Hence $F(g)$

is the identity. This implies that every conic on V_{14} is transformed by g into itself.

By Theorem 1.2, we have 16 different conics $C_1, \dots, C_{16}$ passing through the generic point $x \in V_{14}$, which are transforms of the 16 chords of C' in X. Two different conics C_i, C_j cannot meet at a point y, different from x. Indeed, their proper transforms in X^+ (we are using the notations of Theorem 1.2) are the results l_i^+, l_j^+ of the floppings of two distinct chords l_i, l_j of C'. Two distinct chords of C' are disjoint, because otherwise the 4 points $(l_i \cap l_j) \cap C'$ would be coplanar, which would contradict the linear normality of C'. Hence l_i^+, l_j^+ are disjoint. They meet the exceptional divisor M^+ of $X^+ \longrightarrow V_{14}$ at one point each, hence $C_i \cap C_j = \{x\}$. As $g(C_i) = C_i$ and $g(C_j) = C_j$, this implies $g(x) = x$. This ends the proof. □

5.4. **Two correspondences between $H_{4,0}, H_{5,1}$.** This subsection contains some complementary information on the relations between the families of rational normal quartics and of elliptic normal quintics on X which can be easily deduced from the above results.

For a generic cubic 3-fold X and any point $c \in J^2(X)$, the Abel–Jacobi maps $\Phi_{4,0}, \Phi_{5,1}$ define a correspondence $Z_1(c)$ between $H_{4,0}, H_{5,1}$ with generic fibers over $H_{5,1}$, $H_{4,0}$ of dimensions 3, respectively 5:

$$Z_1(c) = \{(\Gamma, C) \in H_{4,0} \times H_{5,1} \mid \Phi_{4,0}(\Gamma) + \Phi_{5,1}(C) = c\} \ .$$

The structure of the fibers is given by Theorems 3.2 and 5.2: they are, respectiveley, birational to X and isomorphic to $\mathbb{P}^5$.

There is another correspondence, defined in [MT]:

$$Z_2 = \{(\Gamma, C) \in H_{4,0} \times H_{5,1} \mid C + \Gamma = \mathbb{F}_1 \cap X \text{ for a rational normal scroll } \mathbb{F}_1 \subset \mathbb{P}^4\} \ .$$

It is proved in [MT] that the fiber over a generic $C \in H_{5,1}$ is isomorphic to C, and the one over $\Gamma \in H_{4,0}$ is a rational 3-dimensional variety. In fact, we have the following description for the latter:

Lemma 5.5. *For any rational normal quartic $\Gamma \in X$, we have $Z_2(\Gamma) \simeq PGL(2)$.*

Proof. Let $\Gamma \subset \mathbb{P}^4$ be a rational normal quartic. Then there exists a unique $PGL(2)$-orbit $PGL(2) \cdot g \subset PGL(5)$ transforming Γ to the normal form

$$\{(s^4, s^3t, \dots, t^4)\}_{(s:t)\in\mathbb{P}^1} = \{(x_0, \dots, x_4) \mid \mathrm{rk} \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix} \leq 1\}.$$

There is one particularly simple rational normal scroll S containing Γ:

$$S = \{(us^2, ust, ut^2, vs, vt)\} = \{(x_0, \dots, x_4) \mid \mathrm{rk} \begin{pmatrix} x_0 & x_1 & x_3 \\ x_1 & x_2 & x_4 \end{pmatrix} \leq 1\}.$$

Geometrically, S is the union of lines which join the corresponding points of the line $l = \{(0,0,0,s,t)\}$ and of the conic $C^2 = \{(s^2, st, t^2, 0, 0)\}$. Conversely, any rational normal scroll can be obtained in this way from a pair (l, C) whose linear span is the whole $\mathbb{P}^4$. Remark that $(s:t) \mapsto (s:t)$ is the only correspondence from l to C such that the resulting scroll contains Γ.

Now, it is easy to describe all the scrolls containing Γ: they are obtained from S by the action of $PGL(2)$. Each non-identical transformation from $PGL(2)$ leaves invariant Γ, but moves both l and C, and hence moves S. $\square$

As the rational normal scrolls in $\mathbb{P}^4$ are parametrized by a rational variety, the Abel–Jacobi image of $C + \Gamma$ is a constant $c \in J^2(X)$ for all pairs (Γ, C) such that $C \in Z_2(\Gamma)$. Hence we have identically $\Phi_{4,0}(\Gamma) + \Phi_{5,1}(C) = c$ on Z_2, so that $Z_2(\Gamma) \subset Z_1(c)(\Gamma)$.

We can obtain another birational description of $Z_2(\Gamma)$ for generic Γ in applying to all the $C \in Z_2(\Gamma)$ the Tregub–Takeuchi transformation χ, centered at Γ. Let $\xi \in V_{14}$ be the indeterminacy point of χ^{-1}.

Proposition 5.6. *On a generic V_{14}, the family of elliptic quintic curves is irreducible. It is parametrized by an open subset of a component $\mathcal{B}$ of $\mathrm{Hilb}^{5n}_{V_{14}}$ isomorphic to $\mathbb{P}^5$, and all the curves represented by points of $\mathcal{B}$ are l. c. i. of pure dimension 1.*

For any $x \in V_{14}$, the family of curves from $\mathcal{B} = \mathbb{P}^5$ passing through x is a linear 3-dimensional subspace $\mathbb{P}^3_x \subset \mathbb{P}^5$. For generic rational normal quartic Γ as above, χ maps $Z_2(\Gamma)$ birationally onto $\mathbb{P}^3_\xi$.

Proof. Gushel constructs in [G2] for any elliptic quintic curve B on V_{14} a rank two vector bundle $\mathcal{G}$ such that $h^0(\mathcal{G}) = 6$, $\det \mathcal{G} = \mathcal{O}(1)$, $c_2(\mathcal{G}) = B$, and proves that the map from V_{14} to $G = G(2,6)$ given by the sections of $\mathcal{G}$ and composed with the Plücker embedding is the standard embedding of V_{14} into $\mathbb{P}^{14}$. Hence $\mathcal{G}$ is isomorphic to the restriction of the universal rank 2 quotient bundle on G (in particular, it has no moduli), and the zero loci of its sections are precisely the sections of V_{14} by the Schubert varieties $\sigma_{11}(L)$ over all hyperplanes $L \subset \mathbb{C}^6 = H^0(\mathcal{G})^\vee$. These zero loci are l. c. i. of pure dimension 1. Indeed, assume the contrary. Assume that $D = \sigma_{11}(L) \cap V_{14}$ has a component of dimension > 1. Anyway, $\deg D = \deg \sigma_{11}(L) = 5$, hence if $\dim D = 2$, then V_{14} has a divisor of degree $\leq 5 < 14 = \deg V_{14}$. This contradicts the fact that V_{14} has index 1 and Picard number 1.

One cannot have $\dim D > 2$, because otherwise V_{14} would be reducible. Hence $\dim D \leq 1$, and it is l. c. i. of pure dimension 1 as the zero locus of a section of a rank 2 vector bundle. All the zero loci B of sections of $\mathcal{G}$ form a component $\mathcal{B}$ of the Hilbert scheme of V_{14} isomorphic to $\mathbb{P}^5$.

The curves B from $\mathcal{B}$ passing through x are the sections of V_{14} by the Schubert varieties $\sigma_{11}(L)$ for all L containing the 2-plane S_x represented by the point $x \in G(2,6)$, and hence form a linear subspace $\mathbb{P}^3$ in $\mathbb{P}^5$.

Now, let us prove the last assertion. Let $C \in Z_2(\Gamma)$ be generic. We have $(C \cdot \Gamma)_{\mathbb{F}_1} = 7$, therefore the map χ, given by the linear system $|\mathcal{O}(8) - 5\Gamma)|$, sends it to a curve $\tilde{C}$ of degree $8 \cdot 5 - 5 \cdot 7 = 5$. So, the image is a quintic of genus 1. Let $k = \operatorname{mult}_\xi \tilde{C}$. The inverse χ^{-1} being given by the linear system $|\mathcal{O}(2) - 5\xi|$, we have for the degree of $C = \chi^{-1}(\tilde{C})$: $\quad 5 = 2\deg \tilde{C} - 5k = 10 - 5k$, hence $k = 1$, that is, ξ is a simple point of $\tilde{C}$. Thus the generic $C \in Z_2(\Gamma)$ is transformed into a smooth elliptic quintic $\tilde{C} \in V_{14}$ passing through ξ. By the above, such curves form a $\mathbb{P}^3$ in the Hilbert scheme, and this ends the proof. ◻

5.7. **Period map of varieties V_{14}.** We have seen that one can associate to any Fano variety V_{14} a unique cubic 3-fold X, but to any cubic 3-fold X a 5-dimensional family of varieties V_{14}. Now we are going to determine this 5-dimensional family. This will give also some information on the period map of varieties V_{14}. Let $\mathcal{A}_g$ denote the moduli space of principally polarized abelian varieties of dimension g.

Theorem 5.8. *Let $\mathcal{V}_{14}$ be the moduli space of smooth Fano 3-folds of degree 14, and let $\Pi : \mathcal{V}_{14} \to \mathcal{A}_5$ be the period map on $\mathcal{V}_{14}$. Then the image $\Pi(\mathcal{V}_{14})$ coincides with the 10-dimensional locus $\mathcal{J}_5$ of intermediate jacobians of cubic threefolds. The fiber $\Pi^{-1}(J)$, $J \in \mathcal{J}_5$, is isomorphic to the family $\mathcal{V}(X)$ of the V_{14} which are associated to the same cubic threefold X, and birational to $J^2(X)$.*

Proof. For the construction of $\mathcal{V}_{14}$ and for the fact that $\dim \mathcal{V}_{14} = 15$, see Theorem 0.9 in [Mu].

According to Theorem 2.2, there exists a 5-dimensional family of varieties V_{14}, associated to a fixed generic cubic 3-fold X, which is birationally parametrized by the set M of isomorphism classes of vector bundle $\mathcal{E}$ obtained by Serre's construction starting from normal elliptic quintics $C \subset X$. By Corollary 3.3, M is an open subset of $J^2(X)$. Hence all the assertions follow from Proposition 1.7.

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